The Right Honourable Lord Chief Justice Bovill was admitted into the Society.

The following communications were read:

I. "On the Internal Distribution of Matter which shall produce a given Potential at the Surface of a Gravitating Mass." By G. G. Stokes, M.A., Sec.R.S., Lucasian Professor of Mathematics in the University of Cambridge. Received April 18, 1867.

It is known that if either the potential of the attraction of a mass attracting according to the law of the inverse square of the distance, or the normal component of the attraction, be given all over the surface of the mass, or any surface enclosing it (which latter case may be included in the former by regarding the internal density as null between the assumed enclosing surface and the actual surface), the potential and consequently the attraction at all points external to the surface and at the surface itself is determinate. This proposition leads to results of particular interest when applied to the Earth, as I showed in two papers published in 1849*, where among other things I proved that if the surface be assumed to be, in accordance with observation, of the form of an ellipsoid of revolution, Clairaut's Theorem follows independently of the adoption of the hypothesis of original fluidity, or even of that of an internal arrangement in nearly spherical strata of equal density.

But though the law of the variation of gravity which was originally obtained as a consequence of the hypothesis of primitive fluidity, and was afterwards found by Laplace to hold good, on the condition that the surface be an ellipsoid of revolution as well as a surface of equilibrium, provided only the mass be arranged in nearly spherical strata of equal density, be thus proved to be true whatever be the internal distribution, the question may naturally be asked, Does not the condition that the potential at the surface shall have its actual value require that the internal distribution shall be compatible with that of a fluid mass, or at any rate shall be such that the whole mass shall be arranged in nearly spherical strata of equal density? Such a question was in fact asked me by an eminent mathematician at the time to which I have alluded. I replied by referring to the well-known property of a sphere, according to which a central mass may be distributed uniformly over its surface without affecting the external attraction, by applying which proposition to a mass such as the Earth we may evidently, without affecting the external attraction, leave a large excentrically situated cavity

absolutely vacuous, the matter previously within it having been distributed outside it. It is known further that the mass of a particle may be distributed over any surface whatsoever enclosing the particle without affecting the external attraction, and in this way we see at once that we may leave any internal space we please, however excentrically situated, wholly vacuous; nor is it necessary in doing so to introduce an infinite density, by distributing the whole mass previously within that space over its surface, since that mass may be conceived to be divided into an infinite number of infinitely small parts, which are respectively distributed over an infinite number of surfaces surrounding the space in question. These considerations, however, though they readily show that the internal distribution may be widely different from any that is compatible with the hypothesis of primitive fluidity, do not lead to the general expression for the internal density. Circumstances have recently recalled my attention to the subject, and I can now indicate the mode of obtaining the general expression required in the case of any given surface.

Let the mass be referred to the rectangular axes of $x$, $y$, $z$, and let $\rho$ be the density, $V$ be the potential of the attraction. Then for any internal point $V$ satisfies, as is well known, the partial differential equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi\rho, \ldots \ldots (1)$$

or as it may be written for brevity $\nabla V = 0$. This equation may be extended to all space by imagining the body continued infinitely, but having a density which is null outside the limits of the actual body; and by adopting this convention we need not trouble ourselves about those limits. Conversely, if $V$ be a continuously varying function of $x$, $y$, $z$, which vanishes at an infinite distance, and satisfies the partial differential equation (1), $V$ is the potential of the attraction of the mass whose density at the point $(x, y, z)$ is $\rho$; or, in other words,

$$V = \iiint \frac{\rho}{r} \, dx' \, dy' \, dz', \ldots \ldots \ldots (2)$$

where $r$ is the distance between the points $(x, y, z)$ and $(x', y', z')$, $\rho'$ the density at $(x', y', z')$, and the limits are $-\infty$ to $+\infty$, is the complete integral of (1) subject to the condition that $V$ shall vanish at an infinite distance.

This may be proved in different ways; most directly perhaps by taking the expression for the potential ($U$ suppose) which forms the right-hand member of (2), substituting for $\rho'$ its equivalent $-\frac{1}{4\pi} \nabla V'$, $V'$ being the same function of $x', y', z'$ that $V$ is of $x, y, z$, and transforming the integral in the manner done by Green*, when we readily find $U = V$.

Suppose now that we have a given closed surface $S$ containing within it all the attracting matter, and that the potential has a given, in general variable, value $V_0$ at the surface. For the portion of space external to $S$, $V$ is to be determined by the general equation $\nabla V = 0$, subject to the conditions $V = V_0$ at the surface, and $V = 0$ at an infinite distance. We know that the problem of determining $V$ under these circumstances admits of one and but one solution, though it is only for a very limited number of forms of the surface $S$ that the solution can actually be effected. Conceive the problem, however, solved, and from the solution let the value of $\frac{dV}{dv}$ at the surface be found, $v$ being measured outwards along the normal. Now complete $V$ for infinite space by assigning to the space within $S$ any arbitrary but continuous* function we please, subject to the two conditions, 1st, that at the surface it is equal to the given function $V_0$; 2ndly, that it gives for the value of $\frac{dV}{dv}$ at the surface that already got from the solution of the problem referred to in this paragraph. This of course may be done in an infinite number of ways, just as we may in an infinite number of ways join two points in a plane by a continuous curve starting from the two points respectively in given directions, which curve may be either expressed by some algebraical or transcendental equation, or conceived as drawn liberâ manu, and thought of independently of any idea of algebraical expression. The function $V$ having been thus assigned to the space internal to $S$, the equation (1) gives, according to what we have seen, the most general expression for the density of the internal matter.

There is, however, no distinction made in this between positive and negative matter, and if we wish to avoid introducing negative matter we must restrict the function $V$ for the space internal to $S$ to satisfy the imparity

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} > 0.$$  

It is easy from the general expression to show, what is already known, that the matter may be distributed in an infinitely thin, and consequently infinitely dense stratum over the surface $S$, and that such a distribution is determinate.

We know that there exists one and but one continuous function applying to the space within $S$ which satisfies the equation $\nabla V = 0$, and is equal to

* To avoid prolixity, I include in "continuous" the requirement that the differential coefficients of the function, to any order required, shall vary continuously. What that order may be it is perfectly easy in any case to see. We may of course imagine distributions in which the density becomes infinite at one or more points, lines, or surfaces, but so that a finite volume contains only a finite mass. But such distributions may be regarded as limiting, and therefore particular, cases of a distribution in which the density is finite; and therefore the supposition that $\xi$ is finite, does not in effect limit the generality of our results.
\[ V_0 \]\ at the surface. Call this function \( V_1 \). It is to be remarked that the value of \( \frac{dV}{dv} \) at the surface is not the same as that of \( \frac{dV}{dv} \), \( V \) being the external potential, though \( V_1 \) and \( V \) are there each equal to \( V_0 \). The argument, it is to be observed, does not assume that the two are different; it merely avoids assuming that they are the same; the result will prove that they cannot be the same all over \( S \) unless the density, and consequently the potential, be everywhere null, and therefore \( V_0 = 0 \). Now attribute to the interior of \( S \) a function \( V \) which is equal to \( V_1 \) except over a narrow stratum adjacent to \( S \), the thickness of which will in the end be supposed to vanish, within which \( V \) is made to deviate from \( V_1 \) in such a manner as to render the variation of \( \frac{dV}{dv} \) continuous and rapid instead of abrupt.

On applying equation (1), we see that the density is everywhere null except within this stratum, in which it is very great, and in the limit infinite. For the total quantity of matter contained in any portion of the stratum, we have from (1)

\[
\frac{1}{4\pi} \iiint \nabla V \, dx \, dy \, dz,
\]

the integration extending over that portion. Let the portion in question be that corresponding to a very small area \( A \) of the surface \( S \); we may suppose it bounded laterally by the ultimately cylindrical surface generated by a normal to \( S \) which travels round the perimeter of \( A \). Taking now rectangular coordinates \( \lambda, \mu, \rho \), of which the last is parallel to the normal at one point of \( A \), since \( V \) is not changed in form by referring it to a new set of rectangular axes, we have for the mass required

\[
-\frac{1}{4\pi} \iiint \left\{ \frac{d^2V}{d\lambda^2} + \frac{d^2V}{d\mu^2} + \frac{d^2V}{dv^2} \right\} \, d\lambda \, d\mu \, dv.
\]

Of the differential coefficients within brackets, the last alone becomes infinite when the thickness of the stratum, and consequently the range of integration relatively to \( \lambda \), becomes infinitely small. We have in the limit

\[
\int \frac{d^2V}{dv^2} \, dv = \frac{dV}{dv} \cdot \frac{dV_1}{dv},
\]

both differential coefficients having their values belonging to the surface. Hence we have ultimately for the mass

\[
\frac{A}{4\pi} \left( \frac{dV_1}{dv} - \frac{dV}{dv} \right).
\]

Hence, if \( \omega \) be the superficial density, defined as the limit of the mass corresponding to any small portion of the surface divided by the area of that portion,

\[
\omega = \frac{1}{4\pi} \left( \frac{dV_1}{dv} - \frac{dV}{dv} \right), \quad \ldots \quad \ldots \quad \ldots \quad (3)
\]

which is the known expression.
In assigning arbitrarily a function $V$ to the interior of $S$, in order to get the internal density by the application of the formula (1), we may if we please discard the second of the conditions which $V$ had to satisfy at the surface, namely, that $\frac{dV}{dr} = \frac{dV}{d\rho}$; but in that case to the mass, of finite density, determined by (1) must be added an infinitely dense and infinitely thin stratum extending over the surface, the finite superficial density of this stratum being given by (3).

We have seen that the determination of the most general internal arrangement requires the solution of the problem, To determine the potential for space external to $S$, supposed free from attracting matter, in terms of the given potential at the surface; and the determination of that particular arrangement in which the matter is wholly distributed over the surface, requires further the solution of the same problem for space internal to $S$. If, however, instead of having merely the potential given at the surface $S$ we had given a particular arrangement of matter within $S$, and sought the most general rearrangement which should not alter the potential at $S$, there would have been no preliminary problem to solve, since $V$, and therefore its differential coefficients, are known for space generally, and therefore for the surface $S$, being expressed by triple integrals.

Instead of having the attracting matter contained within a closed surface $S$, and the attraction considered for space external to $S$, it might have been the reverse, and the same methods would still have been applicable. The problem in this form is more interesting with reference to electricity than gravitation.

II. "On the Integrability of certain Partial Differential Equations proposed by Mr. Airy." By R. Moon, M.A., late Fellow of Queen's College, Cambridge. Communicated by Professor J. J. Sylvester. Received April 30, 1867.

(Abstract.)

The equation

$$0 = \frac{d^2z}{dy^2} - \alpha \frac{d^2z}{dx^2} + \beta \frac{dz}{dy} + \gamma z, \quad \ldots \ldots \ldots \ldots (1)$$

where $\alpha, \beta, \gamma$ are functions of $x$, includes two equations recently proposed for solution by Mr. Airy, and affords a good illustration of the ordinary incapacity of partial differential equations of the second order for solutions involving arbitrary functions.

If the above equation admit of an integral solution containing one or more arbitrary functions, it must be capable of being derived from an equation of the form

$$F(xyz) = \phi \{ f(xyz) \}, \quad \ldots \ldots \ldots \ldots (2)$$